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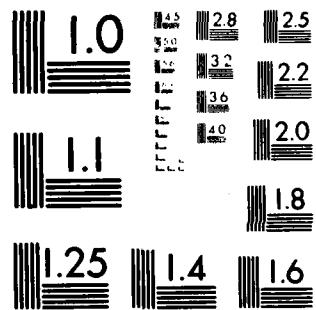
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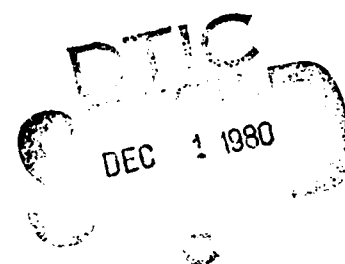
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SOME MATHEMATICAL CONSIDERATIONS IN DEALING WITH  
THE INVERSE PROBLEM

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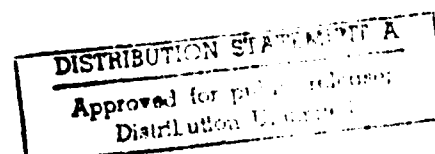


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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  Many problems of mathematical physics can be formulated in terms of the operator equation $Ax = y$ , where $A$ is an integro-differential operator. Given $A$ and $x$ , the solution for $y$ is usually straightforward. However, the inverse problem which consists of the solution for $x$ when given $A$ and $y$ , is much more difficult. In this paper the following questions relative to the inverse problem are explored:			

- (1) Does specification of the operator  $A$  determine the set  $\{y\}$  for which a solution  $x$  is possible?
- (2) Does the inverse problem always have a unique solution?
- (3) Do small perturbations of the forcing function  $y$  always result in small perturbations of the solution?
- (4) What are some of the considerations that enter into the choice of a solution technique for a specific problem?

The concept of an ill-posed problem versus that of a well-posed problem is discussed. Specifically, the manner by which an ill-posed problem may be regularized to a well-posed problem is presented. The concepts are illustrated by several examples.

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Abstract Many problems of mathematical physics can be formulated in terms of the operator equation  $Ax = y$ , where  $A$  is an integro-differential operator. Given  $A$  and  $x$ , the solution for  $y$  is usually straightforward. However, the inverse problem which consists of the solution for  $x$  when given  $A$  and  $y$ , is much more difficult. In this paper the following questions relative to the inverse problem are explored:

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The concept of an ill-posed problem versus that of a well-posed problem is discussed. Specifically, the manner by which an ill-posed problem may be regularized to a well-posed problem is presented. The concepts are illustrated by several examples.

1. Introduction: Many problems of mathematical physics can be formulated in terms of an operator equation

$$Ax = y. \quad (1)$$

Typically,  $y$  denotes the system output and  $x$  denotes the unknown to be solved for. Depending upon the problem, either  $A$  or  $x$  may be defined as the system operator. For example, in a linear system identification problem the goal is to determine the system operator given the response to a specific input. The system operator (i.e. the impulse response) would then be denoted by  $x$  and the input by  $A$ . On the other hand, in a radiation problem the objective might be to solve for the excitation currents on a structure given the radiated field  $y(t)$ . In this case, the excitation current would be denoted by  $x$  and the system operator (i.e. the Green's function operator) would be denoted by  $A$ . Given  $A$  and  $x$ , the direct problem of solving for the output  $y$  is relatively straightforward. However, the inverse problem which consists of the solution for  $x$  given  $A$  and  $y$  is much more difficult. One of the difficulties that may arise is illustrated in Example 1.

Example 1: Consider a linear system with impulse response  $x(t)$  that is excited by an input  $A(t)$  applied at  $t = 0$ . The system response is given by the integral equation

$$y(t) \triangleq Ax = \int_0^t A(t - \tau)x(\tau)d\tau. \quad (2)$$

Given the input  $A(t)$  and the output  $y(t)$ , assume the objective is to find the impulse response  $x(t)$ . In spite of the fact that a solution for  $x(t)$  may exist and be unique, the integral equation of (2) is difficult to solve due to the smoothing action of the convolution operator. For example, denote the exact solution by

$$x_1(t) = x(t) \quad (3)$$

and a perturbed solution by

$$x_2(t) = x(t) + C \sin \omega t \quad (4)$$

Even for very large values of  $C$ , the frequency  $\omega$  can be chosen high enough such that the difference between  $y_1 = Ax_1$  and  $y_2 = Ax_2$  is made to be arbitrarily small. This is demonstrated as follows:

The difference between  $y_2$  and  $y_1$  is

$$y_2 - y_1 = Ax_2 - Ax_1 = C \int_0^t A(t - \tau) \sin \omega \tau \, d\tau. \quad (5)$$

If the input  $A$  is bounded,

$$||A|| \leq M \text{ (a constant)} \quad (6)$$

it follows that

$$y_2 - y_1 \leq CM \int_0^t \sin \omega \tau \, d\tau = \frac{CM(1 - \cos \omega t)}{\omega}. \quad (7)$$

We conclude that

$$||y_2 - y_1|| < \frac{2CM}{\omega}. \quad (8)$$

Obviously, by selecting  $\omega$  to be sufficiently large, the difference  $y_2 - y_1$  can be made arbitrarily small. The ill-posedness of this example is evidenced by the fact that small differences in  $y$  can map into large differences in  $x$ . This is a serious problem because, in practice, measurement of  $y$  will be accompanied by a nonzero measurement error (or representation error in a finite dimensional digital system)  $\delta$ . Use of the "noisy" data can yield a solution significantly different from the desired solution.

When a problem is ill-posed, an attempt should be made to regularize the problem. The solution to the regularized problem will be well-behaved and will offer a reasonable approximation to the solution of the ill-posed problem.

The concepts of well-posed problems and the regularization of ill-posed problems have been discussed in depth in the mathematical literature [1 - 8]. In this paper, we illustrate some of the significant concepts involved by means of simple examples that arise in engineering applications.



## 2. The Definition of a Well-Posed Problem

Hadamard introduced the notion of a well posed (correctly or properly posed) problem in the early 1900's when he studied the Cauchy problem in connection with the solution of Laplace's equation. He observed that the solution  $x(t)$  did not continuously depend on  $y(t)$ . Hadamard concluded that something had to be wrong with the problem formulation because physical solutions did not exhibit this type of discontinuous behavior. Other mathematicians, such as Petrovsky, also reached the same conclusion. As a result of their investigation, a problem characterized by the equation  $Ax = y$  is defined to be well-posed provided the following conditions are satisfied

- (1) The solution  $x$  exists for each element  $y$  in the Hilbert space  $Y$
- (2) The solution  $x$  is unique
- (3) Small perturbations in  $y$  result in small perturbations in the solution  $x$  without the need to impose additional constraints.

If any of the above conditions are violated, the problem is said to be ill-posed. Examples of ill-posed problems are now discussed. We first consider an example in which condition (1) is violated.

Example 2. Suppose it is known that

$$y = x_1 t_1 + x_2 t_2. \quad (9)$$

In addition, assume that the observation  $y$  is contaminated by additive noise. The observations and corresponding time instants are tabulated in Table 1.

Table 1

$t_1$	2	1	1
$t_2$	1	2	3
observation of $y$	3	3	5

Determination of  $x_1$  and  $x_2$  leads to consideration of the following matrix equation:

$$Ax = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \quad (10)$$

Since (10) consists of 3 independent equations in 2 unknowns, a solution does not exist. Condition (1) is violated and the problem is ill-posed.

Example 3: Violation of condition (2) is illustrated in this example. Given a set of measured sample values, consider the problem of using Prony's method to represent a time function  $y(t)$  by a sum of two complex exponentials. Specifically, it is desired to determine  $\beta_i$  and  $C_i$ ,  $i = 1, 2$ , such that

$$C_1 \exp(\beta_1 t) + C_2 \exp(\beta_2 t) = y(t) \quad (11)$$

In this case the unknown vector is

$$x = \begin{bmatrix} \beta_1 \\ \beta_2 \\ C_1 \\ C_2 \end{bmatrix} \quad (12)$$

and  $A$  is the operator that maps  $x$  into  $y$ . Assume

$$y(t) = \exp[-0.00035t] \cos(0.25t). \quad (13)$$

Corresponding to sampling instants given by

$$t = (0, 1, 2, 3, 4) \quad (14)$$

$y(t)$  is given by

$$y = (1, 0.97, 0.88, 0.73, 0.54). \quad (15)$$

However, in practice, measurements of the sample values are accompanied by measurement error. For example, the measured sample values of  $y$  may be given by

$$d = (d_0, d_1, d_2, d_3, d_4) = (1.01, 0.96, 0.89, 0.72, 0.54) \quad (16)$$

Employing Prony's method [9], it is convenient to define

$$s_i = \exp[\beta_i] \quad ; \quad i = 1, 2. \quad (17)$$

Values of  $s_i$  are then obtained from the polynomial equation

$$a_0 s_i^2 + a_1 s_i + a_2 = 0. \quad (18)$$

Using a least-squares error approach [10], the coefficients in (18) are obtained by solving the matrix equation

$$Ca = \begin{bmatrix} c_{00} & c_{10} & c_{20} \\ c_{01} & c_{11} & c_{21} \\ c_{02} & c_{12} & c_{22} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (19)$$

where

$$c_{ij} = \sum_{k=2}^4 d_{k-i} d_{k-j} = c_{ji}. \quad (20)$$

In general, the determinant of C is nonzero and a solution does not exist to (19). However, note that any one of the coefficients in (18) can be set equal to unity without affecting the solution for the roots.

Depending upon whether  $a_0$ ,  $a_1$  or  $a_2$  is chosen to be unity, (19) reduces to one of the following sets of equations:

$$\begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} c_{01} \\ c_{02} \end{bmatrix} ; \quad a_0 = 1 \quad (21a)$$

or

$$\begin{bmatrix} c_{00} & c_{20} \\ c_{02} & c_{22} \end{bmatrix} \begin{bmatrix} a_0 \\ a_2 \end{bmatrix} = - \begin{bmatrix} c_{10} \\ c_{12} \end{bmatrix} ; \quad a_1 = 1 \quad (21b)$$

or

$$\begin{bmatrix} c_{00} & c_{10} \\ c_{01} & c_{11} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = - \begin{bmatrix} c_{20} \\ c_{21} \end{bmatrix} ; \quad a_2 = 1. \quad (21c)$$

Therefore, having set one of the coefficients to unity, the remaining two coefficients may be determined from either (21) a, b or c. These coefficients are then substituted in (18) which is then solved for  $s_1$ . Unfortunately, (21a), b or c do not result in an equivalent set of coefficients. Consequently, the values of  $s_1$  differ as shown in Table 2.

Table 2

<u>Coefficient set to unity</u>	<u>Equation used</u>	<u><math>s_1</math></u>
$a_0 = 1$	(21a)	$0.87 + j0.23$
$a_1 = 1$	(21b)	$0.98 + j0.22$
$a_2 = 1$	(21c)	$1.1 + j0.14$

Since the solution is not unique, condition (2) is violated and the problem is ill-posed when inaccurate measured data is used. Interestingly enough,

had the samples of  $y$  in (15) been available in place of the measured data  $d$  in (16), an equivalent set of coefficients would have been obtained from either (21a),  $b$  or  $c$ . Use of these coefficients in (18) results in  $s_1 = 0.97 \pm j0.25$ , where the numerical values have been rounded off to two decimal places. Therefore, Prony's method is well-posed in the absence of any disturbance.

The final example in this section illustrates a problem for which condition (3) is violated.

Example 4: Consider solution of the matrix equation  $Ax = y$  where  $A$  is the  $4 \times 4$  symmetric matrix of reference [11].

$$A = \begin{bmatrix} 36.86243 & 51.23934 & 53.50338 & 50.49425 \\ 51.23934 & 71.22350 & 74.37005 & 70.18714 \\ 53.50338 & 74.37005 & 77.66275 & 73.29752 \\ 50.49425 & 70.18714 & 73.29752 & 69.17882 \end{bmatrix} \quad (22)$$

and

$$y^T = [192.09940 \quad 267.02003 \quad 278.83370 \quad 263.15773] \quad (23)$$

where the superscript  $T$  denotes transpose. It is readily verified that the exact solution is given by

$$x^T = [1 \quad 1 \quad 1 \quad 1].$$

However, it can be shown that  $A$  is an ill-conditioned matrix in the sense that the ratio of its maximum to minimum eigenvalues is on the order of  $10^{18}$ .

Let us study the effect on the solution  $x$  when only one component of  $y$  is changed in the fifth decimal place. Specifically, we obtain the results presented in Table 3.

Table 3

$y^T \rightarrow$	[192.00939	267.02003	278.83370	263.15773]
$x^T \rightarrow$	[-6,401,472,429	3,866,312,299	1,607,634,613	-953,521,374]
$y^T \rightarrow$	[192.09940	267.02002	278.83370	263.15573]
$x^T \rightarrow$	[3,866,312,299	-2,335,145,694	-970,966,842	575,900,539]
$y^T \rightarrow$	[192.00940	267.02003	278.83369	263.15573]
$x^T \rightarrow$	[1,607,634,615	-970,966,842	-403,733,529	239,462,717]
$y^T \rightarrow$	[192.09940	267.02003	278.83370	263.15772]
$x^T \rightarrow$	[-953,521,374	574,900,539	239,462,717	-142,030,294]

Clearly, extremely small perturbations in  $y$  result in large variations in  $x$ . Condition (3) is violated and the problem is ill-posed.

### 3. Regularization of an ill-posed problem

Most inverse problems of mathematical physics are ill-posed under the three conditions of Hadamard. In a humorous vein Stakgold [12] pointed out that there would likely be a sharp drop in the employment of mathematicians if this were not the case.

Given an ill-posed problem, various schemes are available for defining an associated problem which is well-posed. This approach is referred to as regularization of the ill-posed problem. In particular, an ill-posed problem may be regularized by

- (a) changing the definition of what is meant by an acceptable solution,
- (b) changing the space to which the acceptable solution belongs,
- (c) revising the problem statement,
- (d) introducing regularizing operators, and
- (e) introducing probabilistic concepts so as to obtain a stochastic extension of the original deterministic problem.

The above techniques are now illustrated by a series of examples. Technique (a) is demonstrated in Example (5).

Example 5. Once again consider the problem introduced in Example 2. This resulted in (10) for which a solution did not exist. Nevertheless, an approximate solution is possible. One of several possible approximate solutions for any overdetermined linear system is the Moore Penrose generalized inverse [25] which is a least squares solution to (1) and is given by

$$x = [A^*A]^{-1} A^*y = \begin{bmatrix} 0.8 \\ 1.3142 \end{bmatrix}. \quad (25)$$

the asterisk denotes the transpose conjugate. Note that  $x$  does not exactly satisfy any of the three equations in (10). Yet  $x$  is a reasonable approximate solution. We see that the original problem which was ill-posed because a solution did not exist has been regularized by redefining what is meant by an acceptable solution.

Use of technique (b) is illustrated in Example (6).

Example 6: Relative to Lewis Bojarski inverse scattering [13-15], the relationship at all frequencies and aspects between the back scattered fields  $y(k)$  to the size and shape of the target  $x(r)$  is given by

$$y(k) = \int_{-\infty}^{\infty} x(r) \exp(-jkr) dr. \quad (26)$$

This relation is valid under the assumption that the target currents are obtained using a physical optics approximation. In (26)  $A$  is the Fourier transform operator. The solution to (26) is given by the inverse Fourier transform

$$x(r) = \frac{Q}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(k) \exp[jkr] dk \quad (27)$$

where for convenience, the constant  $Q = 1/\sqrt{2\pi}$  has been introduced.

It is now demonstrated that this problem is ill-posed when solutions are allowed in the metric space for which the norm is defined to be

$$\|x_1(r) - x_0(r)\| \triangleq \max_r |x_1(r) - x_0(r)|. \quad (28)$$

On the other hand, the problem is well posed in the space for which the norm is defined to be

$$\|x_1(r) - x_0(r)\| \triangleq \left\{ \int_{-\infty}^{\infty} [x_1(r) - x_0(r)]^2 dr \right\}^{1/2}. \quad (29)$$

The first norm is referred to as the 'sup' norm whereas the second norm is known as the  $L^2$  norm.

Given a scattered field function  $y_0(k)$ , consider the perturbed scattered field function  $y_1(k)$  such that

$$y_1(k) = y_0(k) + \epsilon \exp[-\alpha k^2]$$

where  $\epsilon$  and  $\alpha$  are arbitrary positive constants. Then

$$\begin{aligned} x_1(r) &= \frac{Q}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[jkr] \{y_0(k) + \epsilon \exp[-\alpha k^2]\} dk \\ &= x_0(r) + \frac{Q\epsilon}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[+jkr - \alpha k^2] dk \\ &= x_0(r) + \frac{Q\epsilon}{\sqrt{2\alpha}} \exp\left[-\frac{r^2}{4\alpha}\right]. \end{aligned} \quad (30)$$

Using the sup norm, the perturbations in the expressions for  $y_1(k)$  and  $x_1(r)$  are bounded by

$$\begin{aligned} \max_k \|y_1(k) - y_0(k)\| &\leq \varepsilon \\ \max_r \|x_1(r) - x_0(r)\| &\leq \frac{Q\varepsilon}{\sqrt{2\alpha}} \end{aligned} \quad (31)$$

Obviously, for a fixed small perturbation in  $y_0(k)$ , the perturbation in  $x_0(r)$  can be arbitrarily large depending upon the choice of  $\alpha$ . Hence, the third condition of Hadamard is violated and the inverse Fourier transform is ill-posed in the metric space using the sup norm. Physically, this implies that the inverse Fourier transform is ill-posed under a pointwise convergence criterion.

On the other hand using the  $\mathcal{L}^2$  norm,

$$\begin{aligned} \|y_1(k) - y_0(k)\| &\triangleq \left\{ \int_{-\infty}^{\infty} [y_1(k) - y_0(k)]^2 dk \right\}^{1/2} = \left\{ \int_{-\infty}^{\infty} \varepsilon^2 \exp[-2\alpha k^2] dk \right\}^{1/2} \\ &= \varepsilon \left( \frac{\pi}{2\alpha} \right)^{1/4} \end{aligned}$$

and

$$\begin{aligned} \|x_1(r) - x_0(r)\| &\triangleq \left\{ \int_{-\infty}^{\infty} [x_1(r) - x_0(r)]^2 dr \right\}^{1/2} = \left\{ \frac{Q^2 \varepsilon^2}{2\alpha} \int_{-\infty}^{\infty} \exp\left[-\frac{r^2}{2\alpha}\right] dr \right\}^{1/2} \\ &= Q\varepsilon \left( \frac{\pi}{2\alpha} \right)^{1/4}. \end{aligned} \quad (32)$$

Since the  $\mathcal{L}^2$  norm of the perturbation for  $x_0(r)$  and  $y_0(k)$  differ only by the constant  $Q$ , small perturbations in  $y_0(k)$  result in small perturbations in  $x_0(r)$ . Hence, the inverse Fourier transform is well posed under the  $\mathcal{L}^2$  norm, which implies that convergence in the mean square for the shape of the structure can be obtained.

In this example, we have illustrated how a problem can be made well-posed just by changing the space of the solution. Physically this implies that the problem is well posed when we want an "average" description of the target and the problem is ill posed when we demand an accurate precise definition of the target.

Example 7: In this example a problem is regularized by means of technique (c) (i.e., revising the problem statement). Consider a linear system identification problem where  $A(t)$ ,  $x(t)$  and  $y(t)$  denote the input, impulse response, and output, respectively. Using the convolution integral the output is

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau)A(\tau)d\tau. \quad (33)$$

Application of Fourier transform theory results in the solution for the impulse response as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{y}(\omega)}{\tilde{A}(\omega)} \exp(j\omega t) d\omega \quad (34)$$

where  $\tilde{y}(\omega)$  and  $\tilde{A}(\omega)$  are the Fourier transforms of  $y(t)$  and  $A(t)$ , respectively.

In practice, the ideal output is contaminated by additive noise. Denote the measured response by  $d(t)$  where

$$d(t) = y(t) + n(t) \quad (35)$$

and  $n(t)$  represents a stationary zero mean additive noise process. The correlation function of  $n(t)$  is denoted by  $\phi(\tau)$  and its Fourier transform by  $\tilde{\phi}(\omega)$ . When the measured noisy response  $d(t)$  is used in place of the ideal output  $y(t)$ , the solution  $x(t)$  is a random process. The variance of  $x(t)$  is given by [16]

$$\sigma_x^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{\phi}(\omega)}{|\tilde{A}(\omega)|^2} d\omega. \quad (36)$$

This problem is considered to be well posed only if  $\sigma_x^2$  is suitably small.

Typically, the noise is assumed to contain a background component of white noise. With this assumption, the power spectral density  $\tilde{\phi}(\omega)$  approaches a nonzero constant  $K$  as  $|\omega|$  approaches infinity. On the other hand, for finite energy inputs,  $|\tilde{A}(\omega)|$  approaches zero as  $|\omega|$  approaches infinity. With these assumptions,  $\sigma_x^2$  is infinite and the problem is obviously ill posed.



This problem can be made well-posed by revising the problem statement. From a mathematical point of view, the input could be restricted to signals whose Fourier transform  $\tilde{A}(\omega)$  is a rational function with numerator polynomial of higher degree than the denominator polynomial. Then, even with a white noise background,  $\sigma_x^2$  remains finite and the problem becomes well-posed. However, this regularization implies the presence in  $A(t)$  of singularity functions such as impulses, doublets, etc. Therefore, the regularization is achieved at the expense of requiring  $A(t)$  to be an unrealizable signal having infinite energy.

A preferable approach is to remove the commonly used assumption of a white noise background. Provided the power spectral density of the noise  $\tilde{\phi}(\omega)$  falls off in frequency at a rate which is at least as fast as that of  $\frac{|\tilde{A}(\omega)|^2}{1+\epsilon}$  ( $\epsilon > 0$ , arbitrary),  $\sigma_x^2$  will be finite. This demonstrates that the noise must be carefully modeled if the problem is to be well-posed.

Technique (d), the application of regularizing operators, is discussed next. In example 2, it was shown that the three simultaneous equations were inconsistent and, therefore, an exact solution did not exist. In example 5, an approximate solution was obtained using the least squares approach. The solution was given by

$$x = [A^*A]^{-1} A^*y . \quad (37)$$

As the dimensionality of the problem is increased, there is a tendency for the solution to oscillate and increase in magnitude. Therefore, the problem becomes ill-posed as the dimensionality increases.

This difficulty may be overcome by introducing regularizing operators which impose additional constraints on the solution. In particular, instead of solving the problem

$$Ax = y \quad (38)$$

attention is focused on the problem of minimizing  $\|Ax - y\|^2$  under the constraint  $\|\hat{L}x\|^2 = C$ , where  $\hat{L}$  is a suitably chosen linear operator. Equivalently, one finds

$$\underset{x}{\text{minimum}} \{ \|Ax - y\|^2 + \mu^2 \|\hat{L}x\|^2 \} \quad (39)$$

where  $\mu$  plays the role of a Lagrange multiplier. If  $\hat{L}$  is the identity

operator, this approach will result in that solution  $x$  which carries out the minimization in (39) having a specified value of  $\|x\|^2$ . If  $\hat{L}$  is the derivative operator, this approach will result in that solution  $x$  which carries out the minimization in (39) having a specified degree of smoothness (i.e. a specified value of  $\|dx/dt\|^2$ ).

The parameter  $\mu$  is determined by the constraint (e.g. the specified value of  $\|x\|^2$  or the specified degree of smoothness or a combination of both) [17].

Many choices of the operator  $\hat{L}$  are possible and one may simultaneously apply several constraints with several corresponding Lagrange multipliers [17].

Solution of the problem posed in (39) is equivalent to the solution of the matrix equation

$$(A^*A + \mu^2 \hat{L}^* \hat{L})x = A^*y \quad (40)$$

and is given by

$$x = (A^*A + \mu^2 \hat{L}^* \hat{L})^{-1} A^*y. \quad (41)$$

This approach is known as the Tykhonov regularizing scheme [3,18]. Note, if  $L$  is the null operator (i.e. if no constraint is imposed), the solution in (41) reduces to the classical least squares solution of (37).

The Tykhonov regularizing scheme has been used to regularize the ill-posed pattern synthesis problem by Deschamps and Cabayan [19] and Mautz and Harrington [20] and the image processing problem [23, 24].

The last approach to be discussed for regularizing an ill-posed problem is technique (e). In this approach a stochastic extension of an otherwise deterministic problem is obtained by introducing probabilistic concepts. Once again, consider the problem of  $Ax = y$ , where  $A$  and  $y$  are deterministic quantities. As mentioned previously, perfect measurement, or observation of  $y$  is impossible. In some cases use of the "noisy" measurements for  $y$  results in an ill-posed problem. With technique (e), the problem is made well-posed by recognizing that uncertainty in the observation of  $y$  causes uncertainty in the resulting solution for  $x$ . Consequently, the solution is viewed as a random process. An error

criterion is specified and a stochastically optimum solution is obtained. The solution is stochastically optimum in the sense that repetition in measurements of  $y$  produces solutions for  $x$  which, on the average, are optimum according to the specified error criterion.

Let the "noisy" measurements of  $y$  be denoted by  $d$  such that

$$d = y + n \quad (42)$$

where  $n$  represents additive noise. In practice,  $n$  may be due to measurement error, numerical roundoff error, or system noise. Substitution of (42) into (38) results in the equivalent problem

$$Ax + n = d. \quad (43)$$

Assume that the means of  $x$ ,  $n$  and  $d$  are known along with the autocorrelation functions  $R_{xx}$ ,  $R_{nn}$ ,  $R_{dd}$  and the cross correlation functions  $R_{xd}$  and  $R_{xn}$ .

Without loss of generality,  $x$ ,  $n$  and  $d$  each can be considered to have zero mean since an equation identical in form to (43) is obtained where the known means are subtracted from the equation. If the solution for  $x$  is denoted by  $\hat{x}$ , the error is given by  $\epsilon = (x - \hat{x})$ . For purposes of illustration, consider the squared-error criterion given by

$$|\epsilon|^2 = |x - \hat{x}|^2. \quad (44)$$

The optimum solution for  $x$  is defined to be that linear solution which minimizes the mean-squared error  $E[|\epsilon|^2]$  where  $E$  denotes the expectation operator. Let the solution for  $x$  have the form

$$\hat{x} = Ld \quad (45)$$

where  $L$  is a linear operator. The mean-squared error is then given by

$$E[|\epsilon|^2] = E[(x - Ld)(x - Ld)^*]. \quad (46)$$

In terms of the known correlation functions the mean-squared error becomes

$$\begin{aligned} E[|\epsilon|^2] &= R_{xx} - LR_{dx} - R_{xd}L^* + LR_{dd}L^* \\ &= R_{xx} - LR_{xd} - R_{xd}L^* + LR_{dd}L^* \\ &= [R_{xx} - R_{xd}R_{dd}^{-1}R_{xd}] + [L - R_{xd}R_{dd}^{-1}]R_{dd}[L - R_{xd}R_{dd}^{-1}]^*. \end{aligned} \quad (47)$$

It is concluded that the minimum mean squared error solution is given by

$$\hat{x} = R_{xd} R_{dd}^{-1} d. \quad (49)$$

Existence and stability of this solution is guaranteed when  $R_{dd}$  is positive definite.

Substituting for  $d$  from (43), the correlation functions in (49) become

$$\begin{aligned} R_{xd} &= E[xd^*] = E[x(x^*A^* + n^*)] = R_{xx}A^* + R_{xn} \\ R_{dd} &= E[dd^*] = E[(Ax + n)(x^*A^* + n^*)] = AR_{xx}A^* + R_{xn}^*A^* + AR_{xn} + R_{nn}. \end{aligned} \quad (50)$$

Thus the solution may also be expressed as

$$\hat{x} = [R_{xx}A^* + R_{xn}][AR_{xx}A^* + R_{xn}^*A^* + AR_{xn} + R_{nn}]^{-1}d. \quad (51)$$

Assuming the random variables  $x$  and  $n$  are uncorrelated,  $R_{xn} = 0$ . The solution then simplifies to

$$\hat{x} = R_{xx}A^*[AR_{xx}A^* + R_{nn}]^{-1}d. \quad (52)$$

Without noise  $R_{nn} = 0$  and the solution reduces to the classical solution  $\hat{x} = A^{-1}d$ .

An obvious difficulty with (52) is that it involves  $R_{xx}$ , the auto-correlation of the deterministic unknown in the original equation (38). Therefore, it is necessary to assume a correlation function for  $x$ . Franklin [21] has shown in a particular application that  $\hat{x}$  is remarkably insensitive to the choice for  $R_{xx}$ .

For the special case in which

$$R_{xx} = \sigma_x^2 I \quad \text{and} \quad R_{nn} = \sigma_n^2 I \quad (53)$$

where  $I$  is the identity operator and the number of unknowns equals the number of equations available, then the solution in (52) becomes [22]

$$\hat{x} = [A^*A + \frac{\sigma_n^2}{\sigma_x^2} I]^{-1} A^*d. \quad (54)$$

With reference to (41), (54) is equivalent to the Tykhonov [4] - Philips [5] - Twomey [6-7] regularization scheme when  $\hat{L}$  is the identity operator. It can also be shown that equivalence between (52) and (41) also holds when  $R_{xx} = (\hat{L}^* \hat{L})^{-1}$  [26].

#### 4. Conclusion:

This paper discusses some mathematical concepts associated with solution of the operator equation  $Ax = y$ . In addition to illustrating, by examples, concepts of ill-posed and well-posed problems, various techniques are presented for regularizing ill-posed problems.

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#### 6. References

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